

THE OVER-RELAXATION FACTOR IN THE NUMERICAL SOLUTION OF THE OMEGA EQUATION¹

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ABSTRACT

Results of tests for the optimum over-relaxation coefficients in the numerical relaxation of the omega equation are presented. One case considers a strong upper-level development for tests on a fixed grid using one-, two-, and three-dimensional forms of the omega equation. In the other case of a "classical storm" the omega equation is relaxed in its three-dimensional form using several different horizontal grids.

For the one- and two-dimensional tests, the relaxation scheme agrees very well with theory. In the three-dimensional tests, the observed over-relaxation coefficients are found to be less than the values given by the theory for all grid sizes considered. A sharp cut-off is found to occur shortly after the optimum over-relaxation value is reached regardless of the number of dimensions of the equation or the size of the grid.

1. INTRODUCTION

The meteorologist is often faced with the problem of solving second-order, linear, partial-differential equations involving 1, 2, or 3 space dimensions. In these days of high-speed computers a solution is normally obtained by one of the various relaxation methods, such as the Richardson or Liebmann method. These techniques are discussed to a point in the literature (Frankel [3], Young [12], Fox [2], Miyakoda [7], and Haltiner et al. [5]). The discussion generally centers around the computer applicability and the speed of convergence of relaxation techniques. The extrapolated Liebmann method is considered more applicable to the computer than the Richardson method, since the former method requires the storage of only one complete set of approximated values. The usefulness of the extrapolated Liebmann method centers around one's ability to obtain an optimum over-relaxation coefficient. Theoretical studies (Frankel [3], Young [12], Miyakoda [7]) yield good estimates of the optimum over-relaxation factor for one- and two-dimensional equations. Young [12] and Miyakoda [7] have tested the three-dimensional equation, but the theory permits only a limited treatment.

In this note we present some results of tests for the optimum over-relaxation coefficients which are used in the numerical relaxation of the omega equation by the extrapolated Liebmann method. The first tests are made on a fixed grid in the study of a strong upper-level develop-

ment along the west coast of the United States. Optimum coefficients are determined for one-, two-, and three-dimensional forms of the omega equation, and these results are compared to the theoretical estimates. In the second set of tests the omega equation is relaxed in its three-dimensional form with several different horizontal grids. The synoptic situation in the latter tests is a "classical storm" in the stage of occluding over the central United States. From these second tests we are able to add to Miyakoda's work for estimating the optimum over-relaxation coefficient when testing with real meteorological equations such as the omega equation.

2. THE OMEGA EQUATION

The omega equation, or the equation for computing the large-scale vertical motion, may be written in general terms as:

$$B(p)\nabla_p^2\omega + \frac{\partial^2\omega}{\partial p^2} = G(x, y, p) \quad (1)$$

where $G(x, y, p)$ is the forcing function and is known from the geopotential data; $\omega \equiv dp/dt$ and is the vertical component of motion in the p -coordinate system; $B(p) = \sigma(p)f_0^{-2}$ is known by specification of the static stability σ as a function of pressure and f_0 , an area-averaged Coriolis parameter. Equation (1) is the final form in the development utilizing the quasi-geostrophic filter and maintaining energy consistency in the alterations of the primitive hydrodynamical equations (Charney [1] and Lorenz [6]).

¹ The information in this note was taken from [8] and [11].

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The numerical solution of the omega equation is accomplished through use of centered finite differences leading to an equation of the general form:

$$\omega_{i,j+1,p} + \omega_{i+1,j,p} + \omega_{i,j-1,p} + \omega_{i-1,j,p} - K'_{ijp}\omega_{ijp} + K_{ijp}[\omega_{i,j,p+\Delta p} + \omega_{i,j,p-\Delta p}] = L_{ijp} \quad (2)$$

where

$$K_{ijp} = \frac{a^2 f_0^2}{m^2 \sigma (\Delta p)^2}; K'_{ijp} = 4 + 2K_{ijp}; L_{ijp} = \frac{a^2 f_0^2}{m^2 \sigma} G_{ijp}.$$

Here a is the horizontal grid interval, Δp is the vertical grid interval, and m is a map projection factor (approximately equal to unity). This form of the difference equation is employed when testing the complete equation (1) or the following two-dimensional equation

$$B(p)\nabla_p^2 \omega = G(x, y, p). \quad (3)$$

In this two-dimensional case equation (2) is modified by setting $K_{ijp} = 0$. When we treat the following one-dimensional equation

$$\frac{\partial^2 \omega}{\partial p^2} = G(x, y, p) \quad (4)$$

the finite difference equation is written as:

$$\omega_{i,j,p+\Delta p} + 2\omega_{ijp} + \omega_{i,j,p-\Delta p} = (\Delta p)^2 G_{ijp} = G'_{ijp}. \quad (5)$$

Hence, whether we are testing equation (2) or (5) we are dealing with a difference equation of the form

$$\Gamma(\omega) = L_{ijp} \quad (6)$$

where the form of the right hand side and of the difference operator are determined by the nature of the equation being handled, i.e., equation (1), (3), or (4). We are primarily concerned with the three-dimensional equation (1), but include tests on the one- and two-dimensional equations (3) and (4) because they can be handled by a simple generalization of our computer program. These latter two equations allow us to verify our technique against the theory for cases in which the theory is complete.

The finite-difference approximation of the omega equation is then solved by the relaxation method. An initial guess of the solution of ω at each grid point is made and the necessary operations, Γ , are performed. The result of this calculation is then compared with the right-hand side for the case of the correct solution. The difference between the actual right-hand side and that computed on the basis of the guess of ω is:

$$R^N = \Gamma(\omega^N) - L_{ijp} \quad (7)$$

where R^N is called the residual and N is the number of scans. The relaxation technique arrives at an approximate solution to ω by the relation:

$$\omega^{N+1} = \omega^N + \alpha R^N \quad (8)$$

α is the over-relaxation factor and is the critical parameter if equation (8) is to work quickly, or at all, for solving equation (1). In practice, we set $\omega^0 = 0$ for the initial scan, $N=0$, and apply equation (8) repeatedly at all grid points where $|R^N| > \epsilon$. Here ϵ is a preset tolerance. This procedure terminates when $|R^N| \leq \epsilon$ at all grid points. Since we apply equation (8) only to points where $|R^N| > \epsilon$, our extrapolated Liebmann relaxation scheme differs slightly from the scheme analyzed theoretically by Young [12] and Miyakoda [7]. For this reason we did not use superscripts of N , $N+1$, etc., on omega in equations (2) and (5). Haltiner et al. [5] varied from the relaxation scheme (8) by introducing a variable in the second term on the right-hand side of equation (8). They state that this Helmholtz-type relaxation appeared to converge more rapidly than our procedure.

The theoretical studies mentioned above give some aid in the determination of the optimum $\alpha(\alpha_{opt})$. These studies show that α_{opt} depends on the number of grid points in each direction (N_x , N_y , N_p), on the nature of the equation itself (i.e., one, two, or three dimensions), on the parameter K_{ijp} (hence on σ , a , f_0 , and Δp in our case), on the type of relaxation scheme used, but not on L_{ijp} (the forcing function) as one might suspect. These studies are quite useful for the one- and two-dimensional equations. However, they treat a three-dimensional equation where $B(p)$ is constant. When dealing with the omega equation (1) the researcher must resort to a sort of trial and error method to determine the optimum α in his case. Guided by Miyakoda's theoretical results and our actual results for an equation with variable $B(p)$, we feel that this trial and error method can be reduced or eliminated.

3. FIXED-GRID STUDIES

The numerical model used in this note yields ω at the four levels of 200 mb., 400 mb., 600 mb., and 800 mb. The input uses contour heights from 10 levels at 100-mb. intervals. The boundary conditions are $\omega=0$ at $p=0$ and 1000 mb., and $\omega=0$ along the horizontal boundaries. The fixed grid used here is a 20×20 array with a 2° latitude grid interval with 6 points along each vertical (i.e., $\Delta p=200$ mb.). The grid is centered about an area along the west coast of the United States encompassing the upper-level development of 1200 gmt, September 30, 1959 (Stuart [10]). The static stability values are those of the standard atmosphere for the four levels of output (see table 1) and $f_0 = 8.9 \times 10^{-5} \text{ sec.}^{-1}$. Even though

TABLE 1.—Values of the static stability, σ .

$$\sigma = -\frac{\alpha}{\theta} \frac{\partial \theta}{\partial p} \text{ in units of m}^3 \text{ cb.}^{-1} \text{ ton}^{-1}$$

p	800 mb.	600 mb.	400 mb.	200 mb.
σ	1.178	2.015	4.252	44.664

initial contour heights are available over an array of 20×20 , the relaxation involves an array of 18×18 at each of the four levels plus the upper and lower boundary points. This yields ω values at 1024 interior grid points.

The experiments in this study with the fixed grid of 2° consisted of solving equations (1), (3), and (4). We will now comment on these results.

Figure 1 is a plot of α versus the number of scans, N , in the relaxation of the three equations (1), (3), and (4). The tolerance for these curves is $\epsilon = 10^{-7}$ cb./sec. for an omega output of the order 10^{-5} cb./sec. The three-dimensional (3-D) curve shows a sharp cut-off near $\alpha = 0.35$. Convergence never was obtained for $\alpha = 0.35$, but the program had been instructed to stop at $N = 200$. Since many points were still to be relaxed, it is felt that the α -curve probably becomes tangent very close to the $\alpha = 0.35$ line. The sharp cut-off shows the importance of such α -studies, since the use of $\alpha = 0.35$ would yield no convergence. The optimum α is approximately 0.32 for $N = 44$. The relatively gentle slope of the α -curve to the left of the optimum values suggests choosing an α -value on that side to cover oneself for slight variations that exist from one case to another. Gates and Riegel [4] also report results showing the sharp transition to non-convergence above the optimum α -value for a similar study.

For the two-dimensional (2-D) curve the optimum α is near 0.44 for $N = 53$ with a cut-off suggested near $\alpha = 0.50$, again stopping the test at $N = 200$. The σ appearing in $B(p)$ was arbitrarily taken as $\sigma = 2$ m.t.s. units for the 2-D relaxation. Because of the smoothness of this α -curve, the choice of α is less critical as long as $\alpha < 0.45$. Again the lower α -values should be favored in choosing the optimum α .

The final curve in figure 1 shows the test for the optimum α using the one-dimensional (1-D) equation (4). The α -curve is very flat and has the optimum $\alpha = 0.65$ for $N = 10$, with a definite cut-off at $\alpha = 1.00$. This shows that the optimum α for the 1-D equation would probably show only slight variations from one grid to another. Values between $\alpha = 0.60$ and 0.70 should be very efficient and well removed from the cut-off zones. Of course a problem as simple as this could probably be done more efficiently by other techniques altogether.

In all the curves of figure 1 we notice that the α -curve tends toward very high N values for low α . This is to be expected since, according to our relaxation scheme (equation (8)), very small adjustments are made when α is small. If $\alpha = 0$, no changes in ω are made regardless of the R^N value and since $\omega^0 = 0$, we obtain convergence only for the trivial case when the actual solution is zero everywhere. It is possible that one could obtain convergence after fewer scans if, depending on the value of N or R , a variable α were used. The rather low values of N for the optimum α make this hardly necessary even when $\omega^0 = 0$. The time spent on each scan is slightly re-

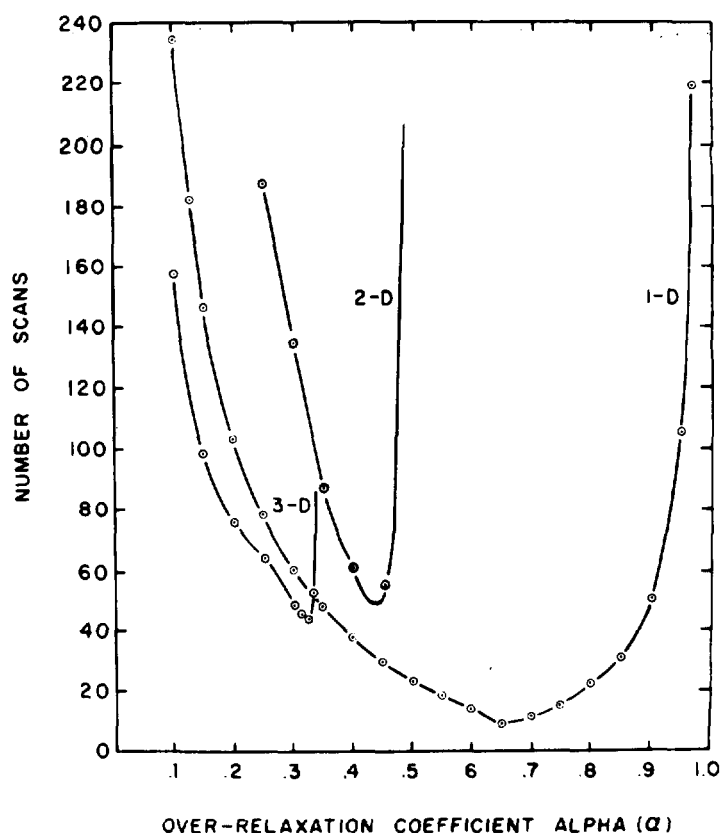


FIGURE 1.—Comparison of optimum α values with number of scans required for convergence. Curves are for relaxation of three-dimensional (3-D), two-dimensional (2-D), and one-dimensional (1-D) forms of the omega equation. Tolerance is $\epsilon = 10^{-7}$ cb./sec.

duced as we approach convergence since ω is altered only for those points where $|R| > \epsilon$.

Table 2 summarizes the comparison of our results for this study, using fixed grids, with the theoretical values. The theoretical optimum α -values were obtained from equations given in Young [12] and Miyakoda [7]. For both the one- and two-dimensional equations, the empirically determined α_{opt} values agree very well with the theory, as do the upper limits on α (i.e., α cut-off). These results tend to suggest that our slight modification of the extrapolated Liebmann relaxation technique has little if any effect on our choice of α_{opt} . The three-dimensional case shows poorer agreement with the theory, but we will discuss those results further in the next section.

TABLE 2.—Summary of the fixed-grid studies, i.e., 2° grid

Equation	α_{opt}		α Cut-Off Observed	Scans N	Grid Points	
	Theory	Observed			$N_x = N_y$	N_p
1-D.....	0.63	0.65	1.00	10	0	6
2-D.....	0.41	0.44	0.50	53	18	0
3-D.....	0.42	0.32	0.35	44	18	6

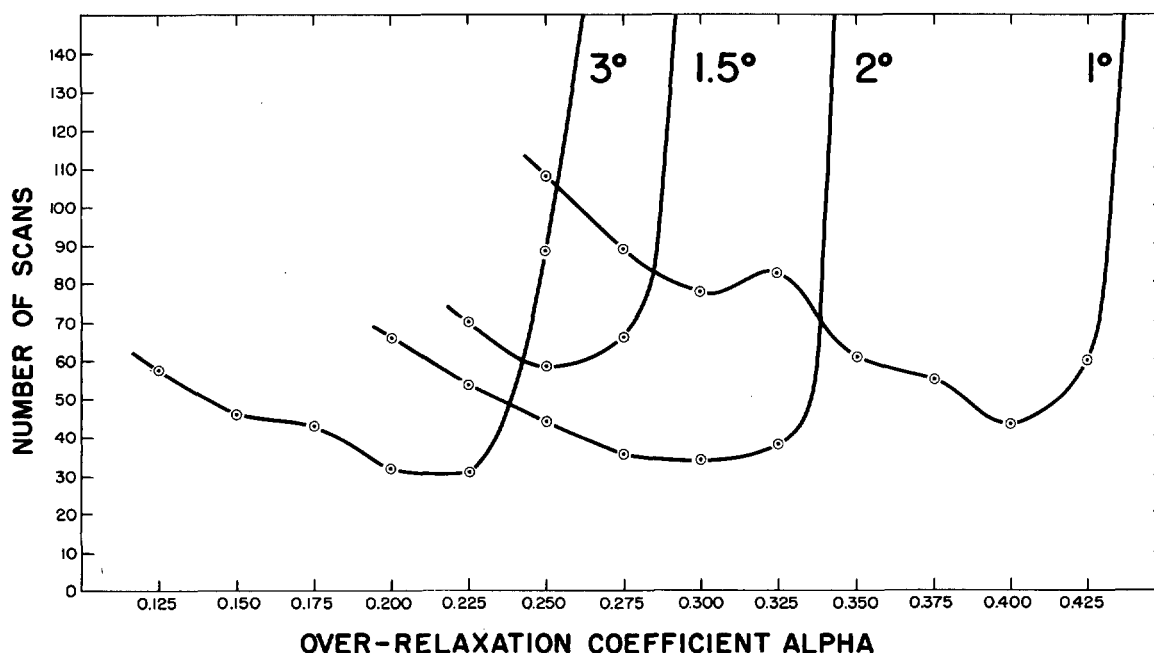


FIGURE 2.—Comparison of the number of scans required for convergence of three-dimensional relaxation with the over-relaxation coefficient, α . Figures along curves indicate in degrees of latitude the grid interval used for the relaxation.

4. VARIABLE-GRID STUDIES

The numerical model and parameters of the previous studies were also used in these studies, but the synoptic situation was a strong cyclogenetic storm of 0000 GMT, November 19, 1957, and the tolerance was $\epsilon = 10^{-6}$ cb./sec. (O'Neill [9]). The experiments using the 3-D equation (1) consist of varying the input of contour height data obtained from four different grids. The input grid area is the same for all sizes, and the grid arrays are 13×13 for 3° , 19×19 for 2° , 25×25 for 1.5° , and 37×37 for 1° latitude interval. However, as above, the relaxation was solved over a grid array, at each of four levels, of 11×11 for 3° , 17×17 for 2° , 23×23 for 1.5° , and 35×35 for 1° , plus the upper and lower boundary points. The same values of f_0 , Δp , and σ were used here as in the previous studies.

Figure 2 shows the results of this test with a comparison of the values obtained from the 1° , 1.5° , 2° , and 3° latitude grids. The variation of the optimum α for the different grid sizes is considerable. The optimum α for a 3° latitude grid occurs at 0.225, although it may also occur between 0.200 and 0.225 because of the small difference of only one scan between these two α -values. The number of scans, N , is 31, which is the smallest number for the optimum α of all computations. The sharp cut-off after the optimum α is reached occurs once again in the results for the 3° latitude grid and also for the other grid sizes. Convergence is not reached with the 3° latitude grid at an $\alpha = 0.275$. No limit was placed on the number of scans,

but a time limit was imposed. For each grid size, the time permitted for the program to run was believed to be sufficient, if convergence were to occur.

The smallest number of scans for the 2° latitude grid is 34, which occurred at the α -value of 0.300. This value is only slightly less than the $\alpha = 0.320$ of the fixed-grid (2° lat.) study. Convergence is obtained in this study at 0.325 and again, as in the previous case, convergence is not reached at the sharp cut-off of $\alpha = 0.350$. The number of scans for the optimum α of the 2° latitude grid is only three more than the number with the 3° latitude grid, but the α -value increases about 35 percent. The α -curve is rather flat about the optimum α , such that in other cases with the 2° latitude grid one should expect an efficient α within the range of values from 0.275 to 0.325.

The optimum α for the 1° latitude grid continues to increase in value (0.400) and in the number of scans ($N = 48$). The three results for 1° , 2° , and 3° latitude grids seem to indicate that over the same area a computation using a large grid distance (small array) would have a smaller optimum α and a smaller number of scans than a computation using a small grid distance (large array). However, the results of the 1.5° latitude grid prove to be quite different. The optimum α is 0.250, which is between the optimum α values of the 2° and 3° latitude grids. Also, the smallest number of scans required for convergence is 59, which is the largest for any of the optimum α results. This behavior of the 1.5° grid results remains unexplained.

TABLE 3.—Summary of variable-grid studies.

Grid Size	α_{opt}		α Cut-Off Observed	Scans N	Grid Points	
	Theory	Observed			$N_x = N_y$	N_p
1°	0.45	0.400	0.450	48	35	6
1.5°	0.44	0.250	0.300	59	23	6
2°	0.42	0.300	0.350	34	17	6
2.5°	0.42	0.320	0.350	44	18	6
3°	0.38	0.225	0.275	31	11	6

*From fixed grid results.

One interesting feature of the plots in figure 2 is the number of scans for an $\alpha=0.325$ for the 1.0° latitude grid. This value is an increase over the previous value of $\alpha=0.300$, and it was expected that a further increase in the number of scans would occur at the next higher α . However, the N for $\alpha=0.350$ took a sharp decrease, and the optimum α did not occur until a value of 0.400 was reached. On a smaller scale this same feature occurs with the 3° lat. grid at $\alpha=0.175$.

Table 3 summarizes the comparison of our results for this study, using variable grids, with the theoretical values. The theoretical estimates for α_{opt} were obtained from an equation given by Miyakoda [7] from the various parameters previously stated, with $\sigma=2$ m.t.s. units (i.e., K_{ijp} is constant in Miyakoda's analysis). As can be seen from table 3, the theoretical values for α_{opt} are all over-estimates as compared to our empirical values. Even more important is that for all grids except the 1° grid our observed α cut-off is less than α_{opt} as given by the simplified theory. Hence the effect of variable σ (i.e., $\sigma=\sigma(p)$) in the factor K_{ijp} appears to be an observed α_{opt} much below the theoretical value, with non-convergence occurring if the α_{opt} is determined by the theory. Even for the 1° grid, the α_{opt} given by the theory might lead to trouble since the observed α cut-off occurs at the same α -value. This above-mentioned shift might also be due to our slightly different relaxation scheme, but we doubt this as a result of the success with the fixed-grid studies.

By inspection of equation (2), we see that K_{ijp} becomes smaller as a and f_0 decrease, or as σ and Δp increase. Hence, by proper adjustment of K_{ijp} , our 3-D equation (2) can become very near to a 2-D equation. In our studies we varied only a , and, indeed, the α_{opt} and α cut-off approached the theory best for the 1° grid case. However, in all of these studies K_{ijp} is very small compared to K'_{ijp} ; hence, all grid sizes behave much like a 2-D equation. It appears inescapable that the variability of K_{ijp} with pressure is responsible for this observed shift of α_{opt} and α cut-off.

5. SUMMARY

Our main conclusions can be stated as follows:

(1) For the 1-D and 2-D equation, our relaxation scheme yields values of α_{opt} that agree well with the theory.

Hence, the theory is a good guide for determining α_{opt} .

(2) With the 3-D equation with coefficients ($B(p)$) which vary in the vertical, the observed α_{opt} is less than the value given by the simple theory. This applied to all grid sizes considered.

(3) The observed sharp cut-off in the α -curve occurs near the α_{opt} value given by the simple theory for the 3-D case. Hence, in solving the omega equation by a relaxation technique one should seek for the α_{opt} at values lower than given by the theory and expect that the α_{opt} given by the theory will lead to non-convergence.

(4) A sharp cut-off will occur shortly after the optimum α -value is reached, regardless of the number of dimensions of the equation in the relaxation or of the size of the grid.

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